On the non-local hydrodynamic-type system and its soliton-like solution

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Abstract We analyze the conditions, which guarantee the existence of periodic and soliton-like traveling wave solutions in the non-local hydrodynamic model of structured media.

1 Introduction

In natural science there exists a number of examples of the creation and stable evolution of so called coherent states, or spatio-temporal patterns [1, 2, 3]. The location and specification of such pattens within the confines of adequate mathematical models is quite difficult, because they are usually described by non-linear partial differential equations. As exceptions, the bi-Hamiltonian equations should be mentioned (see e.g. [4], ch. VII), which can be integrated completely by applying the inverse-scattering method or other associated procedures [5]. Unfortunately, the number of equations which fall under this method is scarce. And, what is possibly more important, coherent structures occur very often in open dissipative systems which cannot be Hamiltonian in principle. In such cases, only a few approaches, the symmetry-based methods [6, 4] among them, could serve as alternatives to numerical modeling. It is worth noticing that in recent years additionally, and seemingly independently to the group methods, the so called anzatz-based approach is widely used [7, 8, 9, 10]. It allows for solutions with pre-set properties (e.g. soliton-like,

kink-like or periodic stationary structures or traveling waves) to be obtained, but, according to our belief, they are less universal than approaches based on self-similarity methods, supported by qualitative analysis. This is because in the latter case one can analyze not only particular self-similar solutions, which fortunately can be expressed analytically, but the whole family with the given symmetry.

Let's now focus our attention on a very important factor which is intensively used in the article. Dissipative models of real processes very often turn out to be close (to some extent) to Hamiltonian models. Such a situation occurs for example in cases when dissipation, relaxation, heterogeneity and similar effects which ruin the regularity of the model, display a small-parameter description. In such a case, while searching for dissipative structures, not only qualitative analysis can be applied, but also one can use the whole power of Hamilton's method. Apart from this, to research a system with small parameters, the ideas and methods of the concept of approximate symmetry can be used, and physically justified small modules, regardless of whether or not they break the symmetry of an unperturbed model, may be taken into consideration.

This article is concerned with the search for wave structures (particularly soliton-like and periodic regimes) in a hydrodynamic model, taking into account the effects of spatio-temporal non-localities. Using the afore-mentioned methods, the criteria for the existence of the wave patterns will be formulated in a wide set of parameter values in those cases where the model under consideration is in some way equivalent to a Hamiltonian one, and in the case, when dissipative effects (associated with relaxation) and the mass forces are taken into account.

The structure of the article is as follows. In section 2 a hydrodynamic-type model of a non-local medium is formulated, and a group-theory reduction to the system of ODEs, describing a family of approximately invariant traveling wave solutions is performed. In section 3 the existence of one-parametric family of periodic and soliton-like solutions is shown when the temporal relaxation and mass forces are ignored. In section 4, based on a generalization of Melnikov's method, and the essential use of the previously obtained exact solution, the existence of soliton-like regimes is demonstrated when small mass force and relaxation effects are incorporated.

2 Nonlocal hydrodynamic-type system and its symmetry reduction

Below we introduce a modelling system, taking into account the non-local effects. These effects are manifested e.g. when an impulse loading is applied to media, possessing an internal structure on mesoscale [11, 12, 13]. Description of the non-linear waves propagation in such media depends in essential way on the ratio of a characteristic size d of elements of the medium structure to a characteristic length λ of the wave pack. If d/λ is $O(1)$, then the basic concepts of continual media are completely false and one should use the description, based, e.g. on the element dynamics methods [14]. The classical continuum mechanics equations applications is justified rather in those cases, when $d/\lambda \ll 1$, and the discreteness of the matter could be completely ignored.

The models studied in this work apply when the ratio d/λ is much less that unity and therefore the continual approach is still valid, but it is not as small that we can ignore the presence of the internal structure. As it have been shown in a number of papers (see e.g. [15]), in the long wave approximation the balance equations for mass and momentum retain their classical form, which in the one-dimensional case can be written as follows:

$$\begin{cases} u_t + p_x = \Im/\rho, \\ \rho_t + \rho^2 u_x = 0. \end{cases} \tag{1}$$

Here u denotes mass velocity, p is pressure, ρ is density, t is time, x is mass (lagrangean) coordinate, $\Im$ is mass force, lower indices denote partial derivatives with respect to subsequent variables. Thus, the whole information about the presence of structure in this approximation is contained in a dynamic equation of state (DES), which should be incorporated to system (1) in order to make it closed.

Generally speaking, DES for multi-component structured media, manifesting the non-local features, takes the form of integral equation [16, 17], linking generalized thermodynamical flow J and generalized thermodynamical force X, causing this flow:

$$J = \int_{-\infty}^{t} \left[\int_{R} K(t,t';\, x,x') X(t',x')\, dx' \right] dt'. \tag{2}$$

Here $K(t,t';\, x,x')$ is a kernel, taking into account nonlocal effects. Function K can be calculated, in principle, by solving dynamic problem of struc-

ture's elements interaction, however such calculations are extremely difficult. Therefore in practice one uses, as a role, some model kernel, describing well enough the main properties of the non-local effects and, in particular, the fact that these effects vanish rapidly as $|t-t'|$ and $|x-x'|$ grow. This property could be used in order to pass from the integral equation (2) to a pure differential equation.

One of the simplest state equations accounting for the effects of spatial nonlocality takes the form:

$$p = \hat{\sigma} \int_{-\infty}^{t} K_1(t,\, t') \left[\int_{-\infty}^{+\infty} K_2\,(x,\, x')\ \rho^n(t', x')\, dx' \right] dt', \tag{3}$$

where $K_2\,(x,\, x') = \exp\left[-(x-x')^2/l^2\right]$ (cf. with [13]). Using the fact that the function $\exp\left[-(x-x')^2/l^2\right]$ extremely quickly approaches zero as $|x-x'|$ grows, we substitute the function $\rho^n(t', x')$ by the first three terms of its decomposition into the power series:

$$\rho^n(t', x') = \rho^n(t', x) + [\rho^n(t', x)]_x \frac{x'-x}{1!} + [\rho^n(t', x)]_{xx} \frac{(x'-x)^2}{2!} + O(|x-x'|^3),$$

obtaining this way the following approximate flow-force relation:

$$p = \hat{\sigma} \int_{-\infty}^{t} K_1\,(t,\, t')\ L\,(\rho,\, \rho_x,\, \rho_{xx})\, dt', \tag{4}$$

where

$$L[\rho,\, \rho_x,\, \rho_{x\,x}] = c_0\, \rho^n(t', x) + c_2\ [\rho^n(t', x)]_{xx}\,,$$

$$c_0 = l \int_{-\infty}^{+\infty} e^{-\tau^2} d\tau = l\sqrt{\pi}, \qquad c_2 = \frac{l^3}{2} \int_{-\infty}^{+\infty} \tau^2 e^{-\tau^2} d\tau = \frac{l^3\sqrt{\pi}}{4}.$$

Now we are going to analyze different function $K_1\,(t,\, t')$, responsible for the relaxing effects inside the elements of the internal structure. A medium with one relaxing component is usually described by the kernel

$$K_1\,(t,\, t') = \hat{\tau}^{-1}\, \exp\left[-\frac{t-t'}{\hat{\tau}} \right]. \tag{5}$$

With such a kernel the flow-force relation (4) can be expressed as a first order PDE. In fact, inserting the kernel (5) into the equation (4) and then differentiating it with respect to t, we get the following DES:

$$\hat{\tau}\, p_t + p = \hat{\sigma}\, L[\rho,\, \rho_x,\, \rho_{x\,x}]. \tag{6}$$

Assuming that

$$K_1\,(t,\,t') = a_1\,\exp\left[-\frac{t-t'}{\tau_1}\right] + a_2\,\exp\left[-\frac{t-t'}{\tau_2}\right]$$

we can obtain the DES for structured media with two relaxing components. In order to obtain a pure differential flow-force relation, we should use p, p_t and p_{tt}. Taking their linear combination with properly chosen coefficients, we obtain the following DES:

$$h\,p_{tt} + \tau\,p_t + p = \alpha\,L[\rho,\,\rho_x,\,\rho_{x\,x}] + \,\mu\,L_t[\rho,\,\rho_x,\,\rho_{x\,x}], \tag{7}$$

where

$$h = \tau_1\,\tau_2,\;\tau = \tau_1 + \tau_2,\;\alpha = \hat{\sigma}\,(a_1\,\tau_1 + a_2\,\tau_2)\,,\;\mu = \hat{\sigma}\;(a_1 + a_2)\,\tau_1\,\tau_2.$$

It is worth noting that a DES formally identical with (7) can be obtained from the equation (3), using in the temporal part of the kernel function

$$K_1\,(t,\,t') = \exp\left[-\frac{t-t'}{\tau_1}\right]\,\cos\left[\frac{t-t'}{\tau_2} + \varphi_0\right],$$

which contains, beside the exponentially decaying, the oscillating term. In fact, using the similar combination as in the previous case, we get the equation, formally identical with the Eq. (7):

$$\bar{h}\,\hat{p}_{tt} + \bar{\tau}\,\hat{p}_t + \hat{p} = \,\bar{\alpha}\,L[\hat{\rho},\,\hat{\rho}_x,\,\hat{\rho}_{x\,x}] + \,\bar{\mu}\,L_t[\hat{\rho},\,\hat{\rho}_x,\,\hat{\rho}_{x\,x}], \tag{8}$$

with

$$\bar{h} = \frac{\tau_1}{2}\,\gamma,\;\bar{\tau} = \gamma,\;\bar{\alpha} = \bar{h}\,\cos\varphi_0\,\gamma,\;\bar{\mu} = \frac{\gamma}{2\,\tau_2}\,(\tau_1\,\cos\varphi_0 - \tau_2\,\sin\varphi_0)\,,$$

where $\gamma = 2\tau_1\,\tau_2^2/\left(\tau_1^2 + \tau_2^2\right)$. In order to maintain the physical meaning of pressure and density we introduce the variables $\hat{p} = p - p_0$, $\hat{\rho} = \rho - \rho_0$, where $p_0 > 0$ and $\rho_0 > 0$ are some constant equilibrium values of the parameters.

The difference between (7) and (8) arises from the fact that the coefficients of these formally coinciding equations belong to distinct domains of the parameter space. For example, the coefficients h and τ from (7) satisfy the relation

$$4\,h\,<\,\tau^2$$

provided that $\tau_1 \neq \tau_2$ while for the $\bar{h}$ and $\bar{\tau}$ the opposite inequality holds. As it is shown in [18, 19], the properties of traveling wave solutions to a non-local hydrodynamic-type model depend in essential way on the values of the parameters.

It is worth noting that DES very similar to (7) have been obtained in [20, 21] on the basis of the phenomenological non-equilibrium thermodynamics formalism. We prefer to present the approach based on the formula (2) and the modeling kernels of non-localities, since it is less cumbersome than that using the thermodynamical methods.

In this work we concentrate on the study of the system of balance equations (1), closed by the DES (6):

$$\begin{cases} u_t + p_x = \Im/\rho, \\ \rho_t + \rho^2 u_x = 0, \\ \hat{\tau} p_t + p = \frac{\beta}{\nu+2}\rho^{\nu+2} + \sigma[\rho^{\nu+1}\rho_{x\,x} + (\nu+1)\rho^{\nu}\,(\rho_x)^2], \end{cases} \tag{9}$$

where $\nu = n-2$, $\quad \beta = c_0\,\hat{\sigma}\,(\nu+2)\,\hat{\tau}$, $\quad \sigma = c_2\,\hat{\sigma}\,(\nu+2)\,\hat{\tau}$.

In what follows we put in (9) $\hat{\tau} = \epsilon\tau$, $\quad \Im/\rho = \epsilon f(\rho)$ with $|\epsilon| \ll 1$. Introduction of a small parameter enables us to employ the approximate symmetry technique and avoid difficulties arising from the fact that the external force, as well as the relaxing terms, can destroy the scaling symmetry of the problem.

It is easy to verify by the straightforward checking, that for $\epsilon = 0$ the system (9) admits the following Lie symmetry group [4, 6] generators:

$$\hat{P}_0 = \frac{\partial}{\partial t}, \quad \hat{P}_1 = \frac{\partial}{\partial x}, \quad \hat{Z} = -\frac{\nu+3}{\nu+1}\,t\,\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial u} + \frac{2}{\nu+1}\,\rho\,\frac{\partial}{\partial \rho} + \frac{4+2\,\nu}{\nu+1}\,p\,\frac{\partial}{\partial p}. \tag{10}$$

For small ϵ, the system (9) admits the operator

$$\hat{X} = \hat{P}_0 + D\,P_1 + \epsilon\,\xi\hat{Z} \tag{11}$$

in the sense of approximate symmetry [22]. Let us note that the first equation of the system (9) admits the operator $\hat{Z}$ in generally accepted sense when $f(\rho) = a\,\rho^{\nu+2}$. Nevertheless the set (10) is not admitted by the system (9), because of the time derivative term contained in the DES.

A passage to the self-similar variables enabling to factorize the system (9), is based on the operator (11). The characteristic system corresponding to this operator, up to $O(\epsilon^2)$, can be presented as follows:

$$\left(1 + \frac{\nu+3}{\nu+1}\,\epsilon\,\xi\,t\right)\,dt = \frac{d\,x}{D} = \frac{d\,u}{\epsilon\,\xi\,u} = \frac{d\,\rho}{\frac{2}{\nu+1}\,\epsilon\,\xi\,\rho} = \frac{d\,p}{\frac{2(2+\nu)}{\nu+1}\,\epsilon\,\xi\,p}. \tag{12}$$

Solving the system (12), and expressing the initial variables in terms of its first integrals, one can construct the following ansatz:

$$u = (1 + \epsilon\,\xi\,t)\,U(\Omega), \quad p = \left(1 + \tfrac{2(2+\nu)}{\nu+1}\,\epsilon\,\xi\,t\right)\Pi(\Omega), \quad \rho = \left(1 + \tfrac{2}{\nu+1}\,\epsilon\,\xi\,t\right)R(\Omega), \tag{13}$$

where

$$\Omega = x - D\,t - \frac{\nu+3}{\nu+1}\,\epsilon\,\xi\,t^2\,D/2$$

is new independent variable.

The approximate self-similar reduction is performed in several steps. Inserting ansatz (13) into the second equation of the system (9), we obtain a first order differential equation admitting the separation of variables:

$$R^2\,\dot{U} - D\,\dot{R} + \frac{2}{\nu+1}\,\epsilon\,\xi\,R = O(\epsilon^2).$$

Integration of this equation gives us the approximate first integral

$$U(\Omega) = C_1 - \frac{D}{R} - \frac{2}{\nu+1}\,\epsilon\,\xi \int \frac{d\,\Omega}{R(\Omega)} + O(\epsilon^2). \tag{14}$$

In what follows we assume that $C_1 = D/R_1$, where $0 < R_1 = const$. For $\epsilon = 0$ such a choice immediately leads to the asymptotics

$$\lim_{x\to+\infty} u(t,\,x) = 0, \qquad \lim_{x\to+\infty} \rho(t,\,x) = R_1. \tag{15}$$

Inserting ansatz (13) into the first equation of the system (9), and using Eq. (14), we obtain the equation

$$\dot{\Pi} - D\left(\frac{D}{R^2}\dot{R} - \frac{2}{\nu} + 1\,\epsilon\frac{\xi}{R}\right) + \epsilon\,\xi\left(C_1 - \frac{D}{R}\right) - \epsilon\,f(R) = O(\epsilon^2). \tag{16}$$

Now we introduce new function:

$$G = \Pi + \frac{D^2}{R}. \tag{17}$$

Taking derivative of (17) with respect to Ω and employing (16), we obtain the equation

$$\dot{G} = \epsilon\left[f(R) - \xi\left(C_1 + \frac{1-\nu}{1+\nu}\cdot\frac{D}{R}\right)\right] + O(\epsilon^2). \tag{18}$$

Inserting ansats (13) into the third equation, we get a second-order ODE. Excluding Π from this equation by means of the formulae (16), (17), and introducing new variable $\dot{R} = Y$ we finally obtain a closed system which, up to $O(\epsilon^2)$, takes the form

$$\dot{R} = Y,$$

$$\sigma\, R^{\nu+2}\, \dot{Y} = G\, R - \left[D^2 + \frac{\beta}{\nu+2} R^{\nu+3} + \sigma(\nu+1) R^{\nu+1} Y^2\right] - \tau\epsilon \frac{D^3}{R}\, Y, \quad (19)$$

$$\dot{G} = \epsilon \left[f(R) - \xi \left(C_1 + \frac{1-\nu}{1+\nu} \frac{D}{R}\right)\right]$$

In the following sections we analyze in detail the system (19) for $\epsilon = 0$ and formulate the conditions of the existence of periodic solutions as well as the homoclinic regimes, corresponding to soliton-like travelling wave solutions of the initial system of PDEs.

3 Qualitative analysis of the reduced system in the case when $\epsilon = 0$

Assuming that $\epsilon = 0$, we immediately get that $G = G_0 = const$, and the system (19) reduces to

$$\begin{cases} \frac{dR}{d\omega} = Y \\ \frac{dY}{d\omega} = \left(\sigma R^{\nu+2}\right)^{-1} \left\{G\, R - \left[D^2 + \frac{\beta}{\nu+2} R^{\nu+3} + \sigma(\nu+1) R^{\nu+1} Y^2\right]\right\}, \end{cases} \quad (20)$$

where $\omega = x - D\, t$. Incorporating the conditions (15), we express G_0 as follows:

$$G = \frac{D^2}{R_1} + \frac{\beta}{\nu+2} R_1^{\nu+2}, \quad (21)$$

Dividing the second equation of the system (20) by the first one, and introducing new variable $Z = Y^2 \equiv (d\,R/d\,\omega)^2$, we get, after some algebraic manipulation, the linear inhomogeneous equation

$$Z'(R) + 2[(\nu+1)\, R]^{-1} Z(R) = 2\left[G\, R - D^2 - \beta R^{\nu+3}/(\nu+2)\right]/(\sigma\, R^{\nu+2}). \quad (22)$$

Solving this equation with respect to $Z = Z(R)$ and next integrating the equation obtained after the substitution $Z = (d\,R/d\,\omega)^2$, we get the following quadrature

$$\omega - \omega_0 = \int \frac{\pm\sqrt{\sigma}\, R^{1+\nu}\, d\,R}{\sqrt{H_1 + 2\,G\,\frac{R^{2+\nu}}{(2+\nu)} - 2\,D^2\frac{R^{1+\nu}}{(1+\nu)} - \beta\,\frac{R^{2(2+\nu)}}{(2+\nu)^2}}}. \qquad (23)$$

The direct analysis of the formula (23) is rather difficult. To realize what sort of solutions we deal with, the methods of qualitative analysis can be applied to the dynamical system (20). It is evident, that all isolated critical points of the system (20) are located on the horizontal axis OR. They are determined by solutions of the algebraic equation

$$P(R) = \frac{\beta}{\nu + 2} R^{\nu+3} - GR + D^2 = 0. \qquad (24)$$

As can be easily seen, one of the roots of equation (24) coincides with R_1. Location of the second real root depends on the relations between the parameters. If $\nu + 3 > 1$, and D^2 satisfies the inequality

$$D^2 > D^2_{cr} = \beta R_1^{\nu+3}, \qquad (25)$$

then there exists the second critical point $R_2 > R_1$. Moreover, if $0 < \nu$ is a natural number, then the polynomial $P(R)$ has the representation

$$P(R) = (R - R_1)(R - R_2)\Psi(R), \qquad (26)$$

where

$$\Psi(R) = \frac{\beta\left[R^{\nu+1} + R^{\nu}(R_2 - R_1) + \cdots + R(R_2^{\nu} - R_1^{\nu}) + (R_2^{\nu+1} - R_1^{\nu+1})\right]}{(\nu+2)(R_2 - R_1)}.$$

Note that $\Psi(R)$ is positive, for $R > 0$. It is a direct consequence of the existence of representation (26) valid for any natural ν. But this is also true for any $\nu > -2$, or, in other words, whenever the function $R^{\nu+3}$ is concave for positive R.

Analysis of system's (20) linearization matrix

$$\hat{M}(R_i,\ 0) = \left[\begin{array}{cc} 0 & 1 \\ \left(\sigma\, R_i^{\nu+2}\right)^{-1} \Psi(R_i)(R_j - R_i) & 0 \end{array}\right], \qquad i = 1, 2, \qquad j \neq i \qquad (27)$$

shows, that the critical points $A_1(R_1, 0)$ is a saddle, while the critical point $A_2(R_2, 0)$ is a center. Thus, the system (20) has only such critical points, which are characteristic to a Hamiltonian system. This circumstance suggests that there could exist a Hamiltonian system equivalent to the system (20). The Hamiltonian function would help us to make a complete study of the phase portrait of system (20), and find out homoclinic trajectories, which correspond to a soliton-like wave packs, providing that such trajectories do exist.

If we introduce a new independent variable T, obeying the equation $\frac{d}{dT} = \sigma R^{\nu+2}\phi(R, Y)\frac{d}{d\omega}$, then we can write down the system (20) as follows:

$$\begin{cases} \frac{dR}{dT} = \sigma R^{\nu+2}\, Y\, \phi(R, Y), \\ \frac{dY}{dT} = \phi(R, Y)\left(GR - \left[D^2 + \frac{\beta}{\nu+2}R^{\nu+3} + \sigma(\nu+1)R^{\nu+1}Y^2\right]\right). \end{cases} \tag{28}$$

Here $\phi(R, Y)$ is a function, that is to be chosen in such a way that the system (28) be Hamiltonian. So we are looking for a function $H(R, Y)$, satisfying the system

$$\frac{\partial H}{\partial Y} = \sigma R^{\nu+2}\phi(R, Y),$$

$$\frac{\partial H}{\partial R} = -\phi(R, Y)\left\{GR - \left[D^2 + \frac{\beta}{\nu+2}R^{\nu+3} + \sigma(\nu+1)R^{\nu+1}Y^2\right]\right\}.$$

Equating mixed derivatives of H, we obtain the characteristic system

$$\frac{dR}{\sigma R^{\nu+2}Y} = \frac{dY}{GR - [D^2 + \frac{\beta}{\nu+2}R^{\nu+3} + \sigma(\nu+1)R^{\nu+1}Y^2]} = \frac{d\phi}{\nu\sigma Y R^{\nu+1}\phi}. \tag{29}$$

The general solution of system (29) can be presented in the form

$$\phi = R^{\nu}\Phi(\rho),$$

where $\Phi(\cdot)$ is an arbitrary function of the variable

$$\rho = R^{\nu+1}\left\{Y^2\Phi + D^2\sigma^{-1}R^{-(\nu+1)}\log\, R + R^2\beta/[\sigma(\nu+2)(\nu+3)] - G\sigma^{-1}R^{-\nu}\right\}.$$

Putting $\phi = 2R^{\nu}$, we can easily restore the Hamiltonian function:

$$H = 2D^2\frac{R^{\nu+1}}{\nu+1} + \frac{\beta}{(\nu+2)^2}R^{2(\nu+2)} + \sigma Y^2 R^{2(\nu+1)} - 2G\frac{R^{\nu+2}}{\nu+2}. \tag{30}$$

As is well-known, the function H is constant on the phase trajectories of the both system (20) and (28), and since the integrating multiplier $\phi = 2R^{\nu}$,

appearing in the formula (28) is positive for $R > 0$, then the phase portraits of the systems (20) and (28) geometrically are identical in the right part of the phase plane $(R,\, Y)$. Thus, all the statements concerning the geometry of the phase trajectories of the system (28) lying in the right half-plane is is valid to the corresponding solutions of the system (20).

Using the linear analysis, we have already shown that the critical point $A_2(R_2,\ 0)$ is a center. Stated above relations between systems (20) and (28) enable to conclude that this point does not change when the nonlinear terms are added. This means that the critical point $A_2(R_2,\ 0)$ is surrounded by the closed trajectories, and, hence, the unperturbed source system (9) possesses a one-parameter family of periodic solutions. If the right branches of the separatrices of the saddle point $A_1(R_1,\ 0)$ go to infinity (the stable branch W^s when $t \to -\infty$ and the unstable branch W^u when $t \to +\infty$), then the domain of finite periodic motions is unlimited. Another possibility is connected with the existence of the limiting trajectory, bi-asymptotic to the saddle. In this case the domain of periodic solutions is bounded, and the source system, in addition to a one-parametric family of periodic solutions, possesses localized soliton-like regimes, corresponding to the homoclinic loop. To answer the question on which of the above mentioned possibilities takes place, the behavior of the saddle separatices, lying to the right from the line $R = R_1$ should be analyzed. We obtain the equation for saddle separatices by putting $H = H(R_1, 0) = H_1$ in the left hand side of the equation (30), and solving it with respect to Y:

$$Y = \pm \frac{\sqrt{H_1 + 2G\frac{R^{\nu+2}}{\nu+2} - [2D^2\frac{R^{\nu+1}}{\nu+1} + \frac{\beta}{(\nu+2)^2}R^{2(\nu+2)}]}}{\sqrt{\sigma}R^{\nu+1}} \tag{31}$$

It is evident from equation (31), that incoming and outgoing separatrices are symmetrical with respect to OR axis. Therefore we can restrict our analysis, e.g., to the upper separatrix Y_+. First of all, let us note, that separatrix Y_+ forms a positive angle with the OR axis in the point$(R_1,\ 0)$:

$$\alpha = \arctan\sqrt{(R_2 - R_1)\Psi(R_1)/\left(\sigma R_1^{\nu+2}\right)}.$$

The above formula arises from the linear analysis of the system (20) in critical point $A_1(R_1,\ 0)$. So $Y_+(R)$ is increasing when $R - R_1$ is small and positive. On the other hand, the function

$$Q(R) = H_1 + 2G\frac{R^{\nu+2}}{\nu+2} - \left[2D^2\frac{R^{\nu+1}}{\nu+1} + \frac{\beta}{(\nu+2)^2}R^{2(\nu+2)}\right],$$

appearing in the RHS of the formula (31), tends to $-\infty$ as $R \to +\infty$, because the coefficient at the highest order monomial $R^{2(\nu+2)}$ is negative, while the index $\nu+2$ is assumed to be positive. Therefore the function $Q(R)$ intersects the open set $R > R_1$ of the OR axis at least once. Let us denote the point of the first intersection by R_3, and let us assume that $R_3 > R_2$. If, with this assumption, we were able to prove that $Y_{\pm}(R)$ form the right angle with the OR axis at the point R_3, then we would have the evidence of tangent intersection of the stable and unstable saddle separatrices.

We begin with the note that $\lim_{R\to R_3^-} Q(R) = +0$. Calculating derivative of $Q(R)$ we get:

$$Q'(R) = -2R^{\nu}\left(\frac{\beta}{\nu+2}R^{\nu+3} - GR + D^2\right) = -2R^{\nu}P(R). \qquad (32)$$

It follows from the decomposition (26), that $Q'(R) < 0$ when $R > R_2$. Therefore

$$\lim_{R\to R_3^-}\frac{dY}{dR} = \frac{RQ'(R) - 2(\nu+1)Q(R)}{2\sqrt{\sigma Q}R^{\nu+2}} = -\infty.$$

So, to complete the prof, we must show that the inequality $R_3 > R_2$ is true. Supposing that the inequalities $R_1 < R_3 < R_2$ take place, we obtain from the equations (32), (26) that $\lim_{R\to R_3^-} Y_+'(R) = +\infty$. On the other hand, the function $Y_+(R)$ approaches zero remaining positive as $R \to R_3^-$. But such behavior is impossible for any function, being regular inside the interval $(R_1,\ R_3)$. The case $R_3 = R_2$ should also be excluded, because the critical point $A_2(R_2, 0)$ is a center. The result obtained can be formulated as follows.

Theorem. *If $\nu > -2$ and $D^2 > \beta\, R_1^{\nu+3}$, then the system (20) possesses a one parameter family of periodic solutions, localized around the critical point $A_2\,(R_2,\ 0)$ in a bounded set* **M**. *The boundary of this set is formed by the homoclinic intersection of separatrices of the saddle point $A_1\,(R_1,\ 0)$.*

Thus the unperturbed source system (9) possesses periodic and soliton-like invariant solutions. Let us note in conclusion that for some special case the integral at the RHS of the formula (23) or, what is the same, at the RHS of the formula

$$T - T_0 = \pm\int\frac{d\,R}{2\,\sqrt{\sigma}\,R^{1+\nu}\,\sqrt{H_1 + 2\,G\,\frac{R^{2+\nu}}{(2+\nu)} - 2\,D^2\frac{R^{1+\nu}}{(1+\nu)} - \beta\,\frac{R^{2(2+\nu)}}{(2+\nu)^2}}}, \qquad (33)$$

can be calculated explicitly:

$$T = \pm\frac{1}{7\sqrt{2}}\left\{7\,\ln\left[\frac{R-1}{3-R+\sqrt{7-2\,R-R^2}}\right]+2\,\sqrt{7}\ln\left[\frac{7-R+\sqrt{7}\sqrt{7-2\,R-R^2}}{R}\right]\right\}. \tag{34}$$

This solution corresponds to the following values of the parameters: $D = 1 = R_1 = \sigma = 1$, $\beta = 1/2$, $\nu = 0$, $G = 5/4$, and $T_0 = 0$.

4 Homoclinic solutions of perturbed system

The purpose of this section is to analyze whether and when the perturbed system (19) possesses the homoclinic solution. Our analysis is based on the generalized Melnikov theory presented in [23]. The theory is based in essential way on the knowledge of the homoclinic solution of unperturbed system, which enables to measure the distance between the stable and unstable saddle separatrices for small ϵ. For this reason we specify the parameters in such a way that they fit the exact solution (34).

Since we are going to use the formalism developed in [23], we should pass in (19) to the independent variable T, for which the unperturbed system becomes Hamiltonian. The standard representation for our system in this case will be as follows:

$$\begin{aligned} \tfrac{d\,X}{d\,T} &= J\,D_X\,H(X,\,G) + \epsilon F(X,\,G), \\ \tfrac{d\,G}{d\,T} &= \epsilon L(X,\,G), \end{aligned} \tag{35}$$

where $X = colon\,(R,\,Y)$, $D_X = colon\left\{\frac{\partial}{\partial\,R},\ \frac{\partial}{\partial\,Y}\right\}$, $0 < \epsilon \ll 1$,

$$F(X,\,G) = colon(-2\,D^3\,Y/R,\,0),$$

$$L(X,\,G) = 2\,\left[f(R) - \xi\,\left(C_1 + \frac{1-\nu}{1+\nu}\cdot\frac{D}{R}\right)\right]$$

while

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In what follows we put $f(R) = a\,R^2 + b$.

So we are going to measure the distance between the saddle separatrices of the perturbed system (35). The problem we deal with differs from the classical one [24, 25] since in the unperturbed case the Hamiltonian (30) depends

on an extra variable G, which plays the role of a parameter. We assume that G belongs to an open set U, containing $G_0 = D^2/R_1 + \beta\, R_1^{\nu+2}/(\nu+2)$.

Following [23], we introduce the set

$$\begin{aligned}\mathcal{M} = \{(X,\, G) \in \mathbb{R}^2 \times \mathbb{R}^1 :\; X = \gamma(G),\ \text{where}\ \gamma(G)\ \text{solves} \\ D_X\, H(\gamma(G),\, G) = 0\ \text{and}\ \det[D_X^2 H(\gamma(G), G)] \neq 0, \forall G \in U\}. \quad (36)\end{aligned}$$

We denote by $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ the 1+1-dimensional stable and unstable manifolds of $\mathcal{M}$. It is obvious that the manifolds $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ coincide as $\epsilon = 0$ (in this case $W^s(\mathcal{M}) = W^u(\mathcal{M}) = \Gamma$). For sufficiently small ϵ, $\mathcal{M}$ is transformed into the locally invariant manifold $W^\varepsilon(\mathcal{M})$. Up to $O(\varepsilon^2)$, the dynamics on $W^\varepsilon(\mathcal{M})$ is governed by the equation

$$\dot{G} = \varepsilon L(\gamma(G), G). \quad (37)$$

For $b = 2\,\xi - a$, $W^\epsilon(\mathcal{M})$ possesses the equilibrium point $(1,\, 0,\, 5/4)$. Analysis of the linear part of equation (37) shows, that this point is stable as $a > \xi/2$ and unstable as $a < \xi/2$.

For nonzero ϵ, the invariant manifolds $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ are transformed into the (locally) invariant manifolds $W^s_\epsilon(\mathcal{M})$ and $W^u_\epsilon(\mathcal{M})$, which do not coincide. Yet at certain conditions the manifolds can still have the points of intersection, different from $W^\varepsilon(\mathcal{M})$. To state these conditions, a distance separating $W^s_\epsilon(\mathcal{M})$ and $W^u_\epsilon(\mathcal{M})$ is measured in vicinity of a point $p \in \Gamma$, lying in the section $G_1 = 5/4$. Up to $O(\varepsilon)$, this distance is equal to the Melnikov integral ([23], Ch. IV):

$$\begin{aligned}M^{G_1}(a,\xi) = \int_{\Gamma(G)} \Big[\langle D_X H, F(X,G)\rangle + \\ + \langle D_X H, (D_G J D_X H) \int L(X,\, G) dT\rangle\Big](X^{G_1}(T), G_1, T) dT, \quad (38)\end{aligned}$$

where $X^{G_1}(T) = (R^{G_1}(T), Y^{G_1}(T))$ is the homoclinic trajectory of the unperturbed system on the $G = G_1$ level corresponding to the hyperbolic fixed point of the vector field on $\mathcal{M}_\varepsilon$. The dependence on $(X^{G_1}(T), G_1, T)$ means in other words that the integration is carried out along the unperturbed trajectory Γ. Formula (34) describes this trajectory as an implicit function $T = T(R^{G_1})$.

Let us begin the calculation of Melnikov integral. Below we present in explicit form the terms from the expression (38):

$$D_X H = \begin{bmatrix} 2Y^2R + 2 + \frac{1}{2}R^3 - 2R\,G \\ 2Y\,R^2 \end{bmatrix},$$

$$JD_X H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, (D_X H) = \begin{bmatrix} 2Y\,R^2 \\ -2Y^2R - 2 - \frac{1}{2}R^3 + 2R\,G \end{bmatrix},$$

$$D_G J D_X H = \begin{bmatrix} 0 \\ 2R \end{bmatrix},$$

$$\int L\,dT = \int 2R^2[a(R^{G_1}(t))^2 + \xi - a - \frac{\xi}{R^{G_1}(t)}]dT.$$

Let's replace T with (34). Then:

$$dT = \frac{1}{2R^2}\sqrt{\frac{R^2}{\frac{7}{8} - 2R - \frac{1}{8}R^4 + \frac{5}{4}R^2}}dR.$$

Finally

$$\int L\,dT = \int (aR^2 + \xi - a - \frac{\xi}{R})\sqrt{\frac{R^2}{\frac{7}{8} - 2R - \frac{1}{8}R^4 + \frac{5}{4}R^2}}dR =$$
$$= \sqrt{2}a\left[1 - R^{G_1}(T)\right]\sqrt{7 - 2R^{G_1}(T) - (R^{G_1}(T))^2} +$$
$$+ 2\sqrt{2}(\xi + 4a)\arcsin\left[\frac{R^{G_1}(T) + 1}{2\sqrt{2}}\right].$$

So the Melnikov integral will be of the form:

$$M^{G_1}(a, \xi) = I_2 - I_1,$$

where

$$I_1 = \int\limits_{\Gamma(G)} 4R^{G_1}(T)\left[Y^{G_1}(T)\right]^2\,dT,$$

$$I_2 = 8\sqrt{2}\int\limits_{\Gamma(G)} R^{G_1}(T)\left[R^{G_1}(T)\right]^3 Y^{G_1}(T)\Big\{\frac{a}{2}\left[1 - R^{G_1}(T)\right]\sqrt{6 - [1 + R^{G_1}(T)]^2} +$$
$$(\xi + 4\,a)\ \arcsin\frac{R^{G_1}(T) + 1}{2\sqrt{2}}\Big\}$$

where $\Gamma(G_1)$ is the unperturbed homoclinic trajectory on the G_1 level.

The computation of I_1 is based on the Green's theorem. As

$$Y^{G_1}(T) = \frac{1}{2(R^{G_1}(T))^2}\frac{dR^{G_1}(T)}{dT} \ ,$$

we have:

$$-I_1 = \int_{\Gamma(G_1)} -4(Y^{G_1}(T))^2 R^{G_1}(T)dT = \oint_{\Gamma(G_1)} -2\frac{Y}{R}dR + 0dY$$

Next,

$$\oint_{\Gamma} -2\frac{Y}{R}dR + 0dY = \int_1^{2\sqrt{2}-1}\int_{-\frac{(R-1)\sqrt{7-2R-R^2}}{2\sqrt{2}R}}^{\frac{(R-1)\sqrt{7-2R-R^2}}{2\sqrt{2}R}} \frac{2}{R}\,dY\,dR =$$

$$= \int_1^{2\sqrt{2}-1}\frac{2(R-1)\sqrt{7-2R-R^2}}{\sqrt{2}R^2}dR = \frac{4}{7}\sqrt{2}(-7+\sqrt{7}\ln(8+3\sqrt{7})) = 0,262789.$$

On calculating the I_2, we use the fact that $Y^{G_1}(T) = 2(R^{G_1}(T))^{-2}\frac{dR^{G_1}(T)}{dT}$:

$$\begin{aligned}
&\textstyle\int_{\Gamma(G_1)} 8\sqrt{2}(R^{G_1}(T))^3 Y^{G_1}(T)(-\frac{1}{2}a(R^{G_1}(T)-1)\sqrt{7-2R^{G_1}(T)-[R^{G_1}(T)]^2}+\\
&\textstyle+(\xi+4a)\arcsin(\frac{R^{G_1}(T)+1}{2\sqrt{2}}))dT =\\
&\textstyle= 2\int_1^{2\sqrt{2}-1} 4\sqrt{2}R(-\frac{1}{2}a(R-1)\sqrt{7-2R-R^2}+(\xi+4a)\arcsin(\frac{R+1}{2\sqrt{2}}))dR =\\
&= -8\sqrt{2}(-3+\pi)a + (4\sqrt{2}+(-8+6\sqrt{2})\pi)(\xi+4a) =\\
&= 2\left[-1,60194a + 7,18141(\xi+4a)\right].
\end{aligned} \tag{39}$$

So, finally

$$M^{G_1}(a,\xi) = 0,262789 - 2\cdot\left[1.60194a + 7,18141(\xi+4a)\right], \tag{40}$$

amd the Melnikov's integral vanishes if

$$\xi = -0,03659 - 3,7769a. \tag{41}$$

Then, for ϵ sufficiently small, stable and unstable manifold intersect near the values of $(a,\ \xi)$, defined by equation (41).

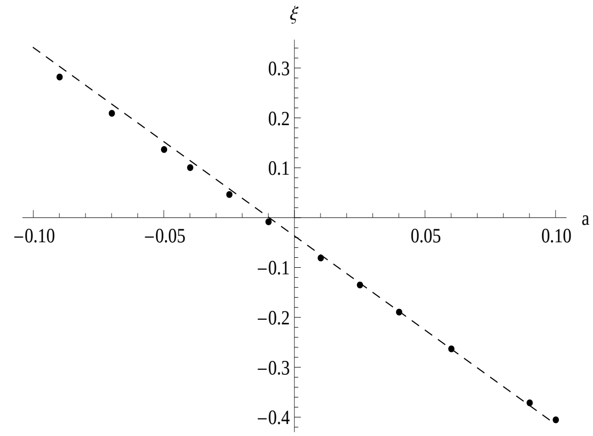

Figure 1: The points (a, ξ), corresponding to the homoclinic intersection (boldface dots) on the background of the straight line (41)

a	-0.09	-0.07	-0.05	-0.04	-0.025	-0.01
δ	-0.0214	-0.0186	-0.01567	-0.014175	-0.0115475	-0.009478
a	0.01	0.025	0.04	0.06	0.08	0.1
δ	-0.0065	-0.0041	-0.0017	0.000125	0.0052	0.0089

The results obtained was verified by direct numerical simulation. To gain the locus of the points in the parameter plane (a, ξ) corresponding to the homoclinic intersection, we used the following modification of the approximate formula (41) obtained with the help of the Melnikov method:

$$\xi = -0,03659 - 3,7769\, a + \delta.$$

The values of the parameters at which the homoclinic intersection is observed are presented in the table shown above. The dynamical system (19) was numerically integrated under the following values of the parameters: $\epsilon = -0.001$, $D = 1 = R_1 = \sigma = 1$, $\beta = 1/2$, $\nu = 0$, $G = 5/4$. The points of the parametric plane (a, ξ) corresponding to the homoclinic intersection are shown in figure 1, together with the straight line given by the formula (41).

5 Concluding remarks

We have considered a non-local hydrodynamic-type model of structured media and studied a family of traveling wave (TW) solutions. Our analysis shows that in the case of a pure spatial non-locality and the absence of mass forces a reduced ODE system is equivalent to a Hamiltonian one. It is worth noting that the initial system of PDEs is not Hamiltonian for the physically justified values of the parameters.

Equivalence of the reduced system to a Hamiltonian one is employed in this work to obtain rigorous proof of the existence of a one–parameter family of periodic and soliton-like TW solutions. At certain values of the parameters these solutions possess an analytic representation, while in general they are described in terms of Jacobi elliptic functions.

Having established a soliton-like solution to the rigorously reducible model, it is possible to analyze the soliton-like solutions' existence in general case, when the exact symmetry of the modeling system has broken down. Using, in this case, the ideas of the approximate symmetry and the generalized Melnikov method, we've stated in explicit form the conditions of the saddle separatrices intersection in cases when the small mass force and the effect of temporal non-locality are incorporated. As there arise from the general considerations presented in [23], intersections of stable and unstable saddle separatrices are not transversal for the given type of perturbation and this is backed by the results of our numerical experiments, that enable us to reveal the loci of the saddle separatrices intersection with no signs of a "homoclinic blow-up" observed.

In view of this, it is interesting to investigate what type of temporal non-locality and external force would lead to the transversal intersection and, as a consequence, to drastic changes in the phase portrait of the reduced system. Our preliminary analysis shows that the transversal intersection can take place in the case of periodic mass force and the DES (7), describing the structured media with two relaxing processes. In fact, the second relaxing process incorporation causes the dynamic equation of state to include as a singular perturbation the second derivative with respect to time. An extra additional variable, playing the same role as G, will then appear in the dynamical system obtained via the approximate symmetry reduction. Let us note in conclusion, that the role of spatial non-locality on the wave patterns' formation and evolution have been discussed in [21]. In particular it was shown in this work that chaotic wave patterns can be gained via homoclinic

bifurcation in the hydrodynamic-type model accounting for the effects of spatial non-localities and where, as in the DES (7), the term p_{tt} is incorporated as a small addend.

References

[1] H. Haken, *Synergetics. An Introduction*, Springer, New York, 1979.

[2] H. Schuster, *Deterministic Chaos*, Wiely, New York, 2005.

[3] M. Rabinovich, D. Trubeckov, *An Introduction in the Theory of vibrations and Waves*, Nauka Publ., Moscow, 1984 (in Russian).

[4] P.Olver: *Applications of Lie Groups to Differential Equations*, Springer, New York, 1993.

[5] R.K.Dodd, J.C.Eilbek, J.D.Gibbon, H.C.Morris: *Solitons and Nonlinear Wave Equations*, Academic Press, London 1984.

[6] L.V. Ovsiannikov: *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[7] E.Fan, *Journ of Physics A: Math and Gen.*, **35**, 6853–6872 (2002).

[8] A. Nikitin, T. Barannyk, *Solitary Waves and Other Solutions for Nonlinear Heat Equations*, arXiv:math-ph/0303004 (2003).

[9] A.Barannyk, I.Yurik, *Proceedings of Institute of Mathematics of NAS of Ukraine*, **50(I)**, 29–33 (2004).

[10] V.A. Vladimirov and E.V. Kutafina, *Rep. Math. Physics*, **54**, 261–271 (2004).

[11] V. Danylenko, V. Sorokina, V. Vladimirov, *Journal of Physics A* , **26** (1993), 7125–7135.

[12] A. Makarenko: *Control and Cybernetics*, **25** 621–630 (1996).

[13] *Peerlings R.H.J. Geers M.G.D., dr Borst R. et al* A critical comparison of nonlocal and gradient-enhanced softening continua//Int. J. of Solids and Structure. – 2001. – Vol. 38. – P. 7723–7746.

[14] Danilenko V.A., Belinskij I.V., Vengrovich D.B., On the Peculiarities of the Wave Processes Connected with the Structure of Geophysical Media, *Doklady Acad. Sci. of Ukraine* **12** (1996) 124–129.

[15] V.A. Vakhnenko and V.Kulich, *Averaged Equations of Wave Dynamics of the Periodic Relaxing Media*, in Boundary Value Problems (Ed. by V.A. Danylenko), Naukova Dumka, Kiev, 1990, 118–120.

[16] V. Ya. Rudyak, *Statistical Theory of Dissipative Processes in Gases and Liquids*, Nauka Publ., Novosibirsk, 1987 (in Russian).

[17] Zubarev D., Tishchenko S., *Physica*, **50** (1972), 285–304.

[18] V. Vladimirov, V. Sidorets, *Proceedings of the Second Int. Conference "Symmetry in Nonlinear Mathematical Physics" (Memorial Prof. W. Fushchych Conference July 7–13 1997, Kyiv, Ukraine), Vol. 2*, Transactions of the Institute of Mathematics NAS of Ukraine, Kyiv, 1997, 409–417.

[19] V. Vladimirov, V. Sidorets, S. Skurativsky, *Rep. Math. Physics*, **44** (1999), 238-246.

[20] V.A. Danylenko and V.A. Vladimirov, *Control and Cybernetics*, **25** (1996), 569-581.

[21] V.A. Danylenko, T.B. Danevych, O.S. Makarenko, S.I. Skurativsky and V.A. Vladimirov, *Self-organization in nonlocal non-equilibrium media*, Naukova Dumka Publ., Kyiv, 2011 (to appear).

[22] N.H. Ibragimov (ed.), *CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 2*, CRC Press, Boca Raton, FL, 1995.

[23] S. Wiggins, *Global Bifurcations and Chaos: Analytical Methods*, Springer, New York, 1988.

[24] V.K. Melnikov, *Trans Moscow Math. Soc.,* **12** (1963), 156-221.

[25] J.Guckenheimer and P.Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer, New York 1987.